J Glob Optim (2006) 36:581–596
DOI 10.1007/s10898-006-9028-x

ORIGINAL PAPER

Generalized *KKM* type theorems in *FC*-spaces with applications (I)

Xie Ping Ding

Received: 13 December 2005 / Accepted: 21 March 2006 /
Published online: 4 July 2006

Abstract The class $KKM(X,Y)$ (resp., s-$KKM(X,Y,Z)$) of set-valued mappings with KKM (resp., s-KKM) property is introduced in FC-spaces without any convexity structure. Some generalized KKM (resp., s-KKM) type theorems are proved in FC-spaces under much weak assumptions. As applications, some new section theorems and coincidence theorems are established in FC-spaces. These theorems generalize many known results in literature. The further applications of these results will be given in a follow-up paper.

Keywords *KKM* (*s-KKM*) type theorem · Transfer compactly open-valued (closed-valued) mapping · Section theorem · Coincidence theorems · *FC*-space

1 Introduction

In 1929, Knaster et al. [1] established the well-known *KKM* theorem in finite dimensional spaces. In 1961, Fan [2] generalized the classical *KKM* theorem to infinite dimensional topological vector spaces. Since then, there exist many generalizations and applications of *KKM* type theorems obtained in underlying spaces (see, e.g. [3–9]).

For a set X, we denote by 2^X and $\langle X\rangle$ the family of all subsets of X and the family of all nonempty finite subsets of X, respectively. If X is a topological vector space and A is a subset of X, we denote by $\mathrm{co}(A)$ and $\overline{A}$ the convex hull of A and the closure of A in X, respectively.

In 1996, Chang and Yen [10] introduced the class $KKM(X,Y)$ of set-valued mappings which is defined as follows:

Definition 1.1 Let X be a convex subset of a topological vector space and Y be a topological space. Let $S,T\colon X\to 2^Y$ be set-valued mappings such that $T(\mathrm{co}(N))\subset S(N)$ for each $N\in\langle X\rangle$, then S is said to be a generalized KKM mapping with respect to

Xie Ping Ding (✉)
College of Mathematics and Software Science, Sichuan Normal University, Chengdu,
Sichuan 610066, P. R. China
e-mail: xieping_ding@hotmail.com

T. A mapping $T: X \to 2^Y$ is said to have KKM property if for each generalized KKM mapping S with respect to T, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property. Write

$$KKM(X,Y) = \{T: X \to 2^Y : T \text{ has } KKM \text{ property}\}.$$

The class $KKM(X,Y)$ include the classes $V(X,Y)$ due to Park et al. [11], $\mathcal{U}_c^k(X,Y)$ due to Park and Kim [12], and $A(X,Y)$ due to Ben-El-Mechaiekh and Deguire [13] as special cases.

Recently Lin et al.[14], Lin and Wang [15] and Lin and Chen [16] further study the class $KKM(X,Y)$ in topological vector spaces. They established some new KKM type theorems, coincidence theorems, fixed point theorems and the equivalent relations between the KKM type theorems and coincidence theorems. As applications, some existence theorems of solutions for generalized vector equilibrium problems are also obtained under suitable assumptions.

In most of known KKM type theorems, the convexity assumptions play a crucial role, which strictly restricts the applicable area of these KKM type theorems. Hence Deng and Xia [9], Ding [7] and Ding et al. [8] established some generalized R-KKM type theorems for generalized R-KKM mappings with compactly closed values and with compactly open values in general topological spaces without any convexity structure, respectively.

In this paper, new classes $KKM(Y,Z)$ and s-$KKM(X,Y,Z)$ of set-valued mappings is introduced in FC-spaces without any convexity structure. Some generalized KKM and s-KKM type theorems for set-valued mappings with transfer compact closed values are established in Finitely Continous (FC)-spaces under much weak assumptions. As applications, some new section theorems and coincidence theorems are obtained in FC-spaces. These theorems unity and generalize many known results in literature. The further applications of our results will be given in a follow-up paper.

2 Preliminaries

Let Δ_n be the standard n-dimensional simplex with vertices $e_0, e_1, \ldots, e_n$. If J is a nonempty subset of $\{0, 1, \ldots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$. For topological space Y, a subset A of Y is said to be compactly open (resp., compactly closed) if for each nonempty compact subset K of Y, $A \bigcap K$ is open (resp., closed) in K. The compact closure and the compact interior of A (see [17]) are defined by

$$\mathrm{ccl}(A) = \bigcap\{B \subset Y : A \subset B \text{ and } B \text{ is compactly closed in } Y\},$$

$$\mathrm{c\,int}(A) = \bigcup\{B \subset Y : B \subset A \text{ and } B \text{ is compactly open in } Y\}.$$

It is easy to see that for each nonempty compact subset K of Y, we have $\mathrm{ccl}(A) \bigcap K = \mathrm{cl}_K(A \bigcap K)$, $\mathrm{c\,int}(A) \bigcap K = \mathrm{int}_K(A \bigcap K)$ and $\mathrm{ccl}(Y \setminus A) = Y \setminus \mathrm{cint}(A)$. A subset A of Y is compactly open (resp., compactly closed) if and only if $\mathrm{cint}(A) = A$ (resp., $\mathrm{ccl}(A) = A$).

Let Y and Z be topological spaces. A set-valued mapping $T: Y \to 2^Z$ is said to be transfer compactly open-valued (resp., transfer compactly closed-valued) on Y (see [17]) if for each $y \in Y$, each nonempty compact subset K of Z and each

$z \in K$, $z \in T(y) \bigcap K$ (resp., $z \notin T(y) \bigcap K$) implies that there exists $y' \in Y$ such that $z \in \inf_K(T(y') \bigcap K)$ (resp., $z \notin \mathrm{cl}_K(T(y') \bigcap K)$). We observe that the notion of transfer compactly open-valued (resp. transfer compactly closed-valued) mappings defined by Lin [18, p. 409] is a special case of the above corresponding notion.

The following result is Lemma 1.1 of Ding [19],

Lemma 2.1 *Let Y and Z be topological spaces and $G: Y \to 2^Z$ a set-valued mapping with nonempty values. Then the following conditions are equivalent:*

(1) *G has the compactly local intersection property,*
(2) *for each compact subset K of Y and for each $z \in Z$, there exists a open subset O_z of Y (which may be empty) such that $O_z \bigcap K \subset G^{-1}(z)$ and $K = \bigcup_{z\in Z}(O_z \bigcap K)$,*
(3) *for any compact subset K of Y, there exists a set-valued mapping $F: Y \to 2^Z$ such that $F(y) \subset G(y)$ for each $y \in Y$, $F^{-1}(z)$ is open in Y and $F^{-1}(z) \bigcap K \subset G^{-1}(z)$ for each $z \in Z$, and $K = \bigcup_{z\in Z}(F^{-1}(z) \bigcap K)$.*
(4) *for each compact subset K of Y and for each $y \in K$, there exists $z \in Z$ such that $y \in \mathrm{int}_K(G^{-1}(z) \bigcap K)$ and*

$$K = \bigcup_{z\in Z}(G^{-1}(z) \bigcap K) = \bigcup_{z\in Z}(\mathrm{c\,int}G^{-1}(z) \bigcap K) = \bigcup_{z\in Z} \mathrm{int}_K(G^{-1}(z) \bigcap K),$$

(5) *$G^{-1}: Z \to 2^Y$ is transfer compactly open-valued on Y.*

Lemma 2.2 *Let Y and Z be topological spaces and $F: Y \to 2^Z$ a set-valued mapping with $Y \neq F^{-1}(z)$ for each $z \in Z$. Then the following conditions are equivalent:*

(1) *F is transfer compactly closed-valued,*
(2) *the mapping $G: Y \to 2^Z$ defined by $G(y) = Z\backslash F(y)$ for each $y \in Y$ is transfer compactly open-valued,*
(3) *for each compact subset K of Z,*

$$\bigcup_{y\in Y}(G(y) \bigcap K) = \bigcup_{y\in Y}(\mathrm{c\,int}G(y) \bigcap K) = \bigcup_{y\in Y} \mathrm{int}_K(K \bigcap G(y)),$$

(4) *for each compact subset K of Z,*

$$\bigcap_{y\in Y}(F(y) \bigcap K) = \bigcap_{y\in Y}(\mathrm{ccl}F(y) \bigcap K) = \bigcap_{y\in Y} \mathrm{cl}_K(K \bigcap F(y)).$$

Proof (1)$\Rightarrow$ (2) For each $y \in Y$, each compact subset K of Z and each $z \in K$, if $z \in K \bigcap G(y) = K \bigcap (Z\backslash F(y))$, then $z \notin K \bigcap F(y)$. By (1), there exists $y' \in Y$ such that $z \notin \mathrm{cl}_K(K \bigcap F(y'))$. Hence $z \in K\backslash \mathrm{cl}_K(K \bigcap F(y')) = \mathrm{int}_K(K \bigcap G(y'))$. Hence G is transfer compactly open-valued. (2)$\Rightarrow$ (1). For each $y \in Y$, each compact subset K of Z and each $z \in K$, if $z \notin K \bigcap F(y)$, then $z \in K\backslash(K \bigcap F(y)) = K \bigcap G(y)$. By (2), there exists $y' \in Y$ such that $z \in \mathrm{int}_K(K \bigcap G(y'))$. Hence $z \notin K\backslash \mathrm{int}_K(K \bigcap G(y')) = \mathrm{cl}_K(K \bigcap F(y'))$. Therefore, F is transfer compactly closed-valued. (2)$\Leftrightarrow$ (3). By (4) and (5) of Lemma 2.1, the equivalent relation holds. (3)$\Rightarrow$ (4). By (3), we have

$$\bigcap_{y\in Y}(F(y)\bigcap K) = K\bigcap\left(\bigcap_{y\in Y}F(y)\right) = K\bigcap\left(\bigcap_{y\in Y}(Z\setminus G(y)\right) = K\bigcap\left(Z\setminus\bigcup_{y\in Y}G(y)\right)$$
$$= K\setminus\left(\bigcup_{y\in Y}\left(G(y)\bigcap K\right)\right) = K\setminus\bigcup_{y\in Y}\mathrm{int}_K\left(K\bigcap G(y)\right)$$
$$= \bigcap_{y\in Y}\left(K\setminus\mathrm{int}_K\left(K\bigcap G(y)\right)\right)$$
$$= \bigcap_{y\in Y}\mathrm{cl}_K\left(K\setminus\left(K\bigcap(Z\setminus F(y))\right)\right) = \bigcap_{y\in Y}\mathrm{cl}_K\left(K\bigcap F(y)\right).$$

Therefore (3)⇒ (4) holds.
(4)⇒ (3). By using similar argument of (3)⇒ (4), we can show that (4)⇒ (3) is also true. This completes the proof.

Remark 2.1 Lemmas 2.1 and 2.2 improve Lemmas 2.1 and 2.5 in [14] respectively.

The following notion was introduced by Ding [20].

Definition 2.1 (Y, φ_N) is said to be a *FC*-space if Y is a topological space and for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ where some elements may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. If A and B are two subsets of Y, B is said to be a *FC*-subspace of Y relative to A if for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ and for any $\{y_{i_0}, \ldots, y_{i_k}\} \subset A\bigcap\{y_0, \ldots, y_n\}$, $\varphi_N(\Delta_k) \subset B$ where $\Delta_k = \mathrm{co}(\{e_{i_0}, \ldots, e_{i_k}\})$. If $A = B$, then B is called a *FC*-subspace of Y.

It is easy to see that the class of *FC*-spaces includes the classes of convex sets in topological vector spaces, *C*-spaces (or *H*-spaces) [21], *G*-convex spaces [12], *L*-convex spaces [22], and many topological spaces with abstract convexity structure as true subclasses. Hence, it is quite reasonable and valuable to study various nonlinear problems in *FC*-spaces.

Definition 2.2 Let (Y, φ_N) be a *FC*-space and Z be a topological space. Let $T, F : Y \to 2^Z$ be two set-valued mappings. F is said to be a generalized *KKM* mapping with respect to T if for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ and each $\{y_{i_0}, \ldots, y_{i_k}\} \subset N$, $T(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^{k} F(y_{i_j})$ where $\Delta_k = \mathrm{co}(\{e_{i_0}, \ldots, e_{i_k})\}$. T is said to have the *KKM* property if for each generalized *KKM* mapping F with respect to T, the family $\{\overline{F(y)} : y \in Y\}$ has the finite intersection property. Write

$$KKM(Y, Z) = \{T\colon\ Y \to 2^Z : T \text{ has the } KKM \text{ property}\}.$$

Clearly, the new class $KKM(Y, Z)$ generalizes the classes $KKM(Y, Z)$ in [10] from convex subsets of topological vector spaces to *FC*-spaces.

Definition 2.3 Let X be a nonempty set, (Y, φ_N) be a *FC*-space and Z be a topological space. Let $s : X \to Y$ be a single-valued mapping, $T : Y \to 2^Z$ and $F : X \to 2^Z$ be two set-valued mappings. F is said to be a generalized *s-KKM* mapping with respect to T if for each $N = \{x_0, \ldots, x_n\} \in \langle X\rangle$ and each $\{x_{i_0}, \ldots, x_{i_k}\} \subset N$,

$T(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^{k} F(x_{i_j})$ where $\varphi_N: \Delta_n \to 2^Y$ is the continuous mapping in touch with $\{s(x_0), \ldots, s(x_n)\} \in \langle Y \rangle$. T is said to have the *s-KKM* property if for each generalized *s-KKM* mapping F with respect to T, the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property. Write

$$s - KKM(X, Y, Z) = \{T: Y \to 2^Z : T \text{ has the } s - KKM \text{ property}\}.$$

The new class *s-KKM*(X, Y, Z) generalizes the class *s-KKM* in [23] from convex subset of topological vector spaces to *FC*-spaces. If $X = Y$ and s is the identity mapping I_X, then *s-KKM*(X, Y, Z) = *KKM*(Y, Z).

By Definitions 2.2 and 2.3, the following result folds.

Lemma 2.3 *If $T \in KKM(Y, Z)$, then for any $s: X \to Y, T \in s\text{-}KKM(X, Y, Z)$.*

Lemma 2.4 *Let (Y, φ_N) be a FC-space, M be a FC-subspace of Y, Z be a topological space and $T \in KKM(Y, Z)$. Then $T|_M \in KKM(M, Z)$.*

Proof Suppose that $S: M \to 2^Z$ is a generalized KKM mapping with respect to $T|_M$, then, for each $N = \{y_0, \ldots, y_n\} \in \langle M \rangle \subset \langle Y \rangle$ and $\{y_{i_0}, \ldots, y_{i_k}\} \subset N$, $T(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^{k} S(y_{i_j})$. Define a set-valued mapping $F: Y \to 2^Z$ by

$$F(y) = \begin{cases} S(y), & \text{if } y \in M, \\ Z, & \text{if } y \in Y \backslash M. \end{cases}$$

It is easy to see that F is a generalized KKM mapping with respect to T. Since $T \in KKM(X, Y)$, the family $\{\overline{F(y)} : y \in Y\}$ has the finite intersection property, which implies the family $\{\overline{S(y)} : y \in M\}$ has the finite intersection property. Hence $T|_M \in KKM(M, Z)$.

Remark 2.2 Lemma 2.4 generalizes Lemma 2.3 of Lin et al. [14] from a convex subset of a topological vector space to a *FC*-space.

Lemma 2.5 *Let (Y, φ_N) be a FC-space, Z be a topological space and $F, G: Y \to 2^Z$ be set-valued mappings. Let $P, H: Z \to 2^Y$ be defined by $P(z) = Y \backslash F^{-1}(z)$ and $H(z) = Y \backslash G^{-1}(z)$ for each $z \in Z$. Then the followings conditions are equivalent:*

(1) *F is a generalized KKM mapping with respect to G,*
(2) *for each $z \in Z$, $H(z)$ is a FC-subspace of Y relative to $P(z)$.*

Proof (1)⇒(2). Suppose (1) is true. If (2) does not hold, then there exist $z \in Z, N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and $\{y_{i_0}, \ldots, y_{i_k}\} \subset N \bigcap P(z)$ such that

$$\varphi_N(\Delta_k) \not\subset H(z) = Y \backslash G^{-1}(z).$$

Hence there exists $y \in \varphi_N(\Delta_k)$ such that $y \in G^{-1}(z)$, i.e., $z \in G(y)$. On the other hand, since $\{y_{i_0}, \ldots, y_{i_k}\} \subset P(z) = Y \backslash F^{-1}(z)$, we have $y_{i_j} \notin F^{-1}(z)$ for all $j = 0, \ldots, k$. It follows that $z \notin \bigcup_{j=0}^{k} F(y_{i_j})$. Hence, we have

$$G(\varphi_N(\Delta_k)) \not\subset \bigcup_{j=0}^{k} F(y_{i_j}),$$

which contradicts (1). Hence (2) must hold.

(2)⇒ (1). Suppose (2) is true. If (1) does not hold, then there exist $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and $\{y_{i_0}, \ldots, y_{i_k}\} \subset N$ such that

$$G(\varphi_N(\Delta_k)) \not\subset \bigcup_{j=0}^{k} F(y_{i_j}).$$

Hence there exist $y \in \varphi_N(\Delta_k)$ and $z \in G(y)$ such that $z \notin \bigcup_{j=0}^{k} F(y_{i_j})$. Hence we have $y_{i_j} \notin F^{-1}(z) = Y \backslash P(z)$ for all $j = 0, \ldots, k$ and so $\{y_{i_0}, \ldots, y_{i_k}\} \subset N \bigcap P(z)$. It follows from (2) that $\varphi_N(\Delta_k) \subset H(z) = Y \backslash G^{-1}(z)$. Therefore, $y \notin G^{-1}(z)$ and $z \notin G(y)$ which is a contradiction. Hence (1) must be true.

Remark 2.3 Lemma 2.5 generalizes Proposition 2 of Lin [24] from G-convex space to FC-space without any convexity structure and the domain and range spaces of mappings may be different.

3 *s-KKM* and *KKM* type theorems

Theorem 3.1 *Let X be a nonempty set, (Y, φ_N) be a FC-space and Z be a topological space. Let $s\colon X \to Y$ be a surjective mapping, $F\colon X \to 2^Z$ be a set-valued mapping and $T \in s\text{-}KKM(X, Y, Z)$ such that*

(1) *$\overline{T(s(X))}$ is compact in Z,*
(2) *F is a generalized s-KKM mapping with respect to T with compactly closed values. Then $\overline{T(s(X))} \bigcap \left(\bigcap_{x \in X} F(x) \right) \neq \emptyset$.*

Proof By (1), $\overline{T(s(X))}$ is compact in Z. Define a set-valued mapping $F^* : X \to 2^{\overline{T(s(X))}}$ by

$$F^*(x) = \overline{T(s(X))} \bigcap F(x), \quad \forall\, x \in X.$$

We claim that F^* is also a generalized s-KKM mapping with respect to T. Since F is a generalized s-KKM mapping with respect to T by (2), for each $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ and $N_1 = \{x_{i_0}, \ldots, x_{i_k}\} \subset \langle N \rangle$, let $y_i = s(x_i),\ i = 0, \ldots, n$, then $M = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and $M_1 = \{y_{i_0}, \ldots, y_{i_k}\} \subset M$ and we have $T(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^{k} F(x_{i_j})$. Since s is surjective, it is clear that $T(\varphi_N(\Delta_k)) \subset \overline{T(s(X))}$. Hence we have

$$T(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^{k} \left(\overline{T(s(X))} \bigcap F(x_{i_j}) \right) = \bigcup_{j=0}^{k} F^*(x_{i_j}).$$

This show that F^* is also a generalized s-KKM mapping with respect to T. Since $T \in s\text{-}KKM(X, Y, Z)$ and $\overline{T(s(X))} \bigcap F(x)$ is closed in $\overline{T(s(X))}$ for each $x \in X$, the family $\{\overline{T(s(X))} \bigcap F(x) : x \in X\}$ has the finitely intersection property. Since $\overline{T(s(X))}$ is compact and F has compactly closed values, Therefore, we have

$$\overline{T(s(X))} \bigcap \left(\bigcap_{x \in X} F(x) \right) \neq \emptyset.$$

Theorem 3.2 *Let X be a nonempty set, (Y, φ_N) be a FC-space and Z be a topological space. Let $s\colon X \to Y$ be a surjective mapping, $F\colon X \to 2^Z$ be a set-valued mapping and $T \in s\text{-}KKM(X, Y, Z)$ such that*

(1) $\overline{T(s(X))}$ *is compact in Z,*
(2) *F is a generalized s-KKM mapping with respect to T with transfer compactly closed values. Then* $\overline{T(s(X))} \bigcap \left(\bigcap_{x\in X} F(x)\right) \neq \emptyset$.

Proof Define a mapping $\mathrm{ccl}\,F\colon X \to 2^Z$ by $(\mathrm{ccl}F)(x) = \mathrm{ccl}F(x)$ for each $x \in X$, it is easy to see that $\mathrm{ccl}F$ is also a generalized *s-KKM* mapping with respect to T with compactly closed values by (2). By Theorem 3.1, $\bigcap_{x\in X}(\overline{T(s(X))} \bigcap \mathrm{ccl}F(x)) \neq \emptyset$. Since $\overline{T(s(X))}$ is compact and F has transfer compactly closed values, by Lemma 2.2, we have

$$\overline{T(s(X))} \bigcap \left(\bigcap_{x\in X} F(x)\right) = \bigcap_{x\in X} \left(\overline{T(s(X))} \bigcap \mathrm{ccl}F(x)\right) \neq \emptyset.$$

Theorem 3.3 *Let (Y, φ_N) be a FC-space and Z be a topological space. Let $F\colon Y \to 2^Z$ be a set-valued mapping and $T \in KKM(Y, Z)$ such that*

(1) *T is a compact mapping,*
(2) *F is a generalized KKM mapping with respect to T with transfer compactly closed values.*

Then $\overline{T(Y)} \bigcap \left(\bigcap_{y\in Y} F(y)\right) \neq \emptyset$.

Proof The conclusion of Theorem 3.3 holds from Theorem 3.2 with $X = Y$ and s being the identity mapping I_X.

Remark 3.1 Theorem 3.3 generalizes Lemma 2.2 and Theorem 2.2 of Lin et al. [14] from topological vector spaces to *FC*-spaces without any convexity structure under much weak assumptions. Theorem 3.2 further generalizes the above results from the class $KKM(Y, Z)$ to the class $s\text{-}KKM(X, Y, Z)$.

Theorem 3.4 *Let $(Y, \{\varphi_N\})$ be a FC-space and Z be a topological space. Let $T \in KKM(Y, Z)$ is a compact mapping and $F, G, M\colon Y \to 2^Z$ be set-valued mappings such that*

(1) *F has transfer compactly closed values,*
(2) *for each $y \in Y$, $G(y) \subset F(y)$ and $T(y) \subset M(y)$,*
(3) *for each $z \in Z$, $Y\backslash M^{-1}(z)$ is a FC-subspace of Y relative to $Y\backslash G^{-1}(z)$.*

Then $\overline{T(Y)} \bigcap \left(\bigcap_{y\in Y} F(y)\right) \neq \emptyset$.

Proof We show that for each $N = \{y_0, \ldots, y_n\}$ and each $\{y_{i_0}, \ldots, y_{i_k}\} \subset N$,

$$T(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^{k} G(y_{i_j}).$$

If it is false, then there exist $N = \{y_0, \ldots, y_n\}$ and $\{y_{i_0}, \ldots, y_{i_k}\} \subset N$ such that

$$T(\varphi_N(\Delta_k)) \not\subset \bigcup_{j=0}^{k} G(y_{i_j}).$$

Hence there exist $\hat{y} \in \varphi_N(\Delta_k)$ and $\hat{z} \in T(\hat{y})$ such that $\hat{z} \notin G(y_{i_j})$ for all $j = 0, \ldots, k$. It follows that

$$\{y_{i_0}, \ldots, y_{i_k}\} \subset N \bigcap (Y \backslash G^{-1}(\hat{z})).$$

By (3), we have

$$\varphi_N(\Delta_k) \subset Y \backslash M^{-1}(\hat{z}).$$

Since $\hat{y} \in \varphi_N(\Delta_k)$, we obtain $\hat{y} \notin M^{-1}(\hat{z})$ and hence $\hat{z} \notin M(\hat{y})$. By (2), $\hat{z} \notin T(\hat{y})$, which contradicts the fact $\hat{z} \in T(\hat{y})$. This show that G is a generalized KKM mapping with respect to T. Since $G(y) \subset F(y)$ for each $y \in Y$ by (2), F is also a generalized KKM mapping with respect to T. All condition of Theorem 3.3 are satisfied. By Theorem 3.3,

$$\overline{T(Y)} \bigcap \left(\bigcap_{y \in Y} F(y) \right) \neq \emptyset.$$

Corollary 3.1 *Let $(Y, \{\varphi_N\})$ be a FC-space and Z be a topological space. Let $T \in KKM(Y, Z)$ is a compact mapping and $F, M \colon Y \to 2^Z$ be set-valued mappings such that*

(1) *F has transfer compactly closed values,*
(2) *for each $y \in Y$, $T(y) \subset M(y)$,*
(3) *for each $z \in Z$, $Y \backslash M^{-1}(z)$ is a FC-subspace of Y relative to $Y \backslash F^{-1}(z)$.*

Then $\overline{T(Y)} \bigcap \left(\bigcap_{y \in Y} F(y) \right) \neq \emptyset$.

Proof The conclusion of Corollary 3.1 holds from Theorem 3.4 with $F = G$.

Corollary 3.2 *Let $(Y, \{\varphi_N\})$ be a FC-space and Z be a topological space. Let $T \in KKM(Y, Z)$ is a compact mapping and $F \colon Y \to 2^Z$ be set-valued mappings such that*

(1) *F has transfer compactly closed values,*
(2) *for each $z \in Z$, $Y \backslash T^{-1}(z)$ is a FC-subspace of Y relative to $Y \backslash F^{-1}(z)$.*

Then $\overline{T(Y)} \bigcap \left(\bigcap_{y \in Y} F(y) \right) \neq \emptyset$.

Proof The conclusion of Corollary 3.2 holds from Corollary 3.1 with $M = T$.

By Lemma 2.5, Theorem 3.4 has the following equivalent form.

Theorem 3.5 *Let $(Y, \{\varphi_N\})$ be a FC-space and Z be a topological space. Let $T \in KKM(Y, Z)$ is a compact mapping and $F, G, M \colon Y \to 2^Z$ be set-valued mappings such that*

(1) *F has transfer compactly closed values,*
(2) *for each $y \in Y$, $G(y) \subset F(y)$ and $T(y) \subset M(y)$,*
(3) *G is a generalized KKM mapping with respect to M.*

Then $\overline{T(Y)} \bigcap \left(\bigcap_{y \in Y} F(y) \right) \neq \emptyset$.

4 Section theorems and coincidence theorems

In this section, by applying our *KKM* type theorems obtained in above section, some new section theorems and coincidence theorems are established in *FC*-spaces.

Theorem 4.1 *Let $(Y, \{\varphi_N\})$ be a FC-space, Z be a topological space and let $T \in KKM(Y, Z)$ is a compact mapping. Let A, B, M be subsets of $Z \times Y$ such that*

(1) *The mapping $P^{-1}: Y \to 2^Z$ is transfer compactly open-valued, where $P: Z \to 2^Y$ is defined by $P(z) = \{y \in Y : (z, y) \notin A\}$ for all $z \in Z$,*
(2) *for all $y \in Y$ and $z \in T(y)$, $(z, y) \in M$ and $B \subset A$,*
(3) *for each $z \in Z$, the set $\{y \in Y : (z, y) \notin M\}$ is a FC-subspace of Y relative to the set $\{y \in Y : (z, y) \notin B\}$.*

Then there exists a point $\hat{z} \in \overline{T(Y)}$ such that $\{\hat{z}\} \times Y \subset A$.

Proof Define set-valued mappings $F: Z \to 2^Y$ and $G, M: Y \to 2^Z$ as follows:

$$F(z) = \{y \in Y : (z, y) \in A\}, \quad \forall\ z \in Z \text{ and}$$

$$G(y) = \{z \in Z : (z, y) \in B\}, \qquad M(y) = \{z \in Z : (z, y) \in M\}, \quad \forall\ y \in Y.$$

Since $F^{-1}(y) = \{z \in Z : (z, y) \in A\} = Z \backslash P^{-1}(y)$, by (1) and Lemma 2.2, $F^{-1}: Y \to 2^Z$ is transfer compactly closed-valued. By (2), $T(y) \subset M(y)$ and $G(y) \subset F^{-1}(y)$ for all $y \in Y$. By the definition of G and M, we have $Y \backslash M^{-1}(z) = \{y \in Y : (z, y) \notin M\}$ and $Y \backslash G^{-1}(z) = \{y \in Y : (z, y) \notin B\}$ for each $z \in Z$. It follows from the condition (3) that for each $z \in Z$, $Y \backslash M^{-1}(z)$ is a *FC*-subspace of Y relative to $Y \backslash G^{-1}(z)$. All conditions of Theorem 3.4 are satisfied. By Theorem 3.4, $\overline{T(Y)} \bigcap (\bigcap_{y \in Y} F^{-1}(y)) \neq \emptyset$. Hence there exists $\hat{z} \in \overline{T(Y)}$ such that $\hat{z} \in F^{-1}(y)$ for all $y \in Y$, i.e. $\{\hat{z}\} \times Y \subset A$. This completes the proof.

Corollary 4.1 *Let $(Y, \{\varphi_N\})$ be a FC-space, Z be a topological space and let $T \in KKM(Y, Z)$ is a compact mapping. Let A, B be subsets of $Z \times Y$ such that*

(1) *The mapping $P^{-1}: Y \to 2^Z$ is transfer compactly open-valued, where $P: Z \to 2^Y$ is defined by $P(z) = \{y \in Y : (z, y) \notin A\}$ for all $z \in Z$,*
(2) *for all $y \in Y$ and $z \in T(y)$, $(z, y) \in B$ and $B \subset A$,*
(3) *for each $z \in Z$, the set $\{y \in Y : (z, y) \notin B\}$ is a FC-subspace of Y.*

Then there exists a point $\hat{z} \in \overline{T(Y)}$ such that $\{\hat{z}\} \times Y \subset A$.

Proof The conclusion of Corollary 4.1 holds from Theorem 4.1 with $M = B$.

Remark 4.1 Corollary 4.1 generalizes Theorem 2.3 of Lin et al. [14] from topological vector spaces to *FC*-space without any convexity structure under much weak assumptions.

Corollary 4.2 *Let $(Y, \{\varphi_N\})$ be a FC-space, Z be a topological space. Let $A \subset Z \times Y$ such that*

(1) *The mapping $P^{-1}: Y \to 2^Z$ is transfer compactly open-valued, where $P: Z \to 2^Y$ is defined by $P(z) = \{y \in Y : (z, y) \notin A\}$ for all $z \in Z$,*
(2) *for each $z \in Z$, the set $\{y \in Y : (z, y) \notin B\}$ is a FC-subspace of Y,*

(3) *there exist a compact subset K of Z and a closed set $B \subset A$ such that for each $y \in Y$, $T(y) = \{z \in K : (y, z) \in B\}$ is nonempty acyclic.*

Then there exists a point $\hat{z} \in \overline{T(Y)}$ such that $\{\hat{z}\} \times Y \subset A$.

Proof Let $(y, z) \in \overline{Gr(T)}$. Then there exists a net $\{(y_\alpha, z_\alpha)\}$ in $Gr(T)$ such that $(y_\alpha, z_\alpha) \to (y, z)$. Hence we have $\{z_\alpha\} \subset K$ and $\{(y_\alpha, z_\alpha)\} \subset B$. Since K is compact and B is closed, we have $z \in K$ and $(y, z) \in B$. This implies that $z \in T(y)$ and $(y, z) \in Gr(T)$. Hence $T : Y \to 2^Z$ has closed graph and $T(Y) \subset K$. By Corollary 3.1.9 of Aubin and Ekeland [25, p.111], T is upper semicontinuous with compact values. By (3), T is an upper semicontinuous set-valued mapping with nonempty compact acyclic values. Hence $T \in V(Y, Z) \subset KKM(Y, Z)$ is a compact mapping. All conditions of Corollary 4.1 are satisfied. The conclusion of Corollary 4.2 holds from Corollary 4.1.

Remark 4.2 Corollary 4.2 generalizes Theorem 2.4 of Lin et al. [14], Theorem 3 of Ha [26] from topological vector spaces to *FC*-spaces without any convexity structure under weaker assumptions.

Corollary 4.3 *Let $(Y, \{\varphi_N\})$ be a FC-space, Z be a topological space and let $T \in KKM(Y, Z)$ is a compact mapping. Let $A \subset Z \times Y$ such that*

(1) *The mapping $P^{-1} : Y \to 2^Z$ is transfer compactly open-valued, where $P : Z \to 2^Y$ is defined by $P(z) = \{y \in Y : (z, y) \notin A\}$ for all $z \in Z$,*
(2) *for all $y \in Y$ and $z \in T(y)$, $(z, y) \in A$,*
(3) *for each $z \in Z$, the set $\{y \in Y : (z, y) \notin A\}$ is a FC-subspace of Y.*

Then there exists a point $\hat{z} \in \overline{T(Y)}$ such that $\{\hat{z}\} \times Y \subset A$.

Proof The conclusion of Corollary 4.3 holds from Corollary 4.1 with $A = B$.

Theorem 4.2 *Let $(Y, \{\varphi_N\})$ be a FC-space and Z be a topological space. Let $T \in KKM(Y, Z)$ be a compact mapping and $F : Y \to 2^Z$, $H, P : Z \to 2^Y$ be set-valued mappings such that*

(1) *F is transfer compactly closed-valued and for each $z \in Z, Y \neq F^{-1}(z)$,*
(2) *for each $y \in Y$, $Z \backslash H^{-1}(y) \subset F(y)$,*
(3) *for each $z \in Z$, $P(z)$ is a FC-subspace of Y relative to $H(z)$.*

Then there exists $(y_0, z_0) \in Y \times Z$ such that $z_0 \in T(y_0)$ and $y_0 \in P(z_0)$.

Proof Define set-valued mappings $G, M : Y \to 2^Z$ by

$$G(y) = Z \backslash H^{-1}(y), \qquad \text{and } M(y) = Z \backslash P^{-1}(y), \quad \forall\, y \in Y.$$

Then, $G(y) \subset F(y)$ for each $y \in Y$ by (2). By the definition of G and M, it is easy to show that $H(z) = Y \backslash G^{-1}(z)$ and $P(z) = Y \backslash M^{-1}(z)$ for each $z \in Z$. By (3), we have that for each $z \in Z$, $Y \backslash M^{-1}(z)$ is a *FC*-subspace of Y relative to $Y \backslash G^{-1}(z)$. Now suppose the conclusion is false, then $T(y) \bigcap P^{-1}(y) = \emptyset$ for all $y \in Y$. Hence for each $y \in Y$, $T(y) \subset Z \backslash P^{-1}(y) = M(y)$. All conditions of Theorem 3.4 are satisfied. By Theorem 3.4, $\overline{T(Y)} \bigcap (\bigcap_{y \in Y} F(y) \neq \emptyset$. Therefore, there exists $z_0 \in \overline{T(Y)}$ such that $z_0 \in F(y)$ for all $y \in Y$, i.e., $Y = F^{-1}(z_0)$, which contradicts (1). Hence there exists $(y_0, z_0) \in Y \times Z$ such that $z_0 \in T(y_0)$ and $y_0 \in P(z_0)$.

Corollary 4.4 *Let $(Y, \{\varphi_N\})$ be a FC-space and Z be a topological space. Let $T \in KKM(Y, Z)$ be a compact mapping and $F: Y \to 2^Z$, $P: Z \to 2^Y$ be set-valued mappings such that*

(1) *F is transfer compactly closed-valued and for each $z \in Z$, $Y \neq F^{-1}(z)$,*
(2) *for each $y \in Y$, $Z \backslash P^{-1}(y) \subset F(y)$,*
(3) *for each $z \in Z$, $P(z)$ is a FC-subspace of Y.*

Then there exists $(y_0, z_0) \in Y \times Z$ such that $z_0 \in T(y_0)$ and $y_0 \in P(z_0)$.

Proof The conclusion of Corollary 4.4 from Theorem 4.2 with $P = H$.

Theorem 4.3 *Let $(Y, \{\varphi_N\})$ be a FC-space and Z be a topological space. Let $T \in KKM(Y, Z)$ be a compact mapping and $H, P, Q: Z \to 2^Y$ be set-valued mappings such that*

(1) *Q^{-1} is transfer compactly open-valued and for each $z \in Z$, $Q(z) \neq \emptyset$,*
(2) *for each $z \in Z$, $Q(z) \subset H(z)$,*
(3) *for each $z \in Z$, $P(z)$ is a FC-subspace of Y relative to $H(z)$.*

Then there exists $(y_0, z_0) \in Y \times Z$ such that $z_0 \in T(y_0)$ and $y_0 \in P(z_0)$.

Proof Define set-valued mapping $F: Y \to 2^Z$ by $F(y) = Z \backslash Q^{-1}(y)$ for all $y \in Y$. By (1) and Lemma 2.2, F is transfer compactly closed-valued and for each $z \in Z$, $Y \neq F^{-1}(z)$ since $F^{-1}(z) = Y \backslash Q(z)$ by the definition of F. The condition (1) of Theorem 4.2 is satisfied. By (2), we have that for each $y \in Y$, $Z \backslash H^{-1}(y) \subset Z \backslash Q^{-1}(y) = F(y)$. The condition (2) of Theorem 4.2 is satisfied. All conditions of Theorem 4.2 are satisfied. By Theorem 4.2, there exists $(y_0, z_0) \in Y \times Z$ such that $z_0 \in T(y_0)$ and $y_0 \in P(z_0)$.

Corollary 4.5 *Let $(Y, \{\varphi_N\})$ be a FC-space and Z be a topological space. Let $T \in KKM(Y, Z)$ be a compact mapping and $P, Q: Z \to 2^Y$ be set-valued mappings such that*

(1) *Q^{-1} is transfer compactly open-valued and for each $z \in Z$, $Q(z) \neq \emptyset$,*
(2) *for each $z \in Z$, $P(z)$ is a FC-subspace of Y relative to $Q(z)$.*

Then there exists $(y_0, z_0) \in Y \times Z$ such that $z_0 \in T(y_0)$ and $y_0 \in P(z_0)$.

Proof The conclusion of Corollary 4.5 holds from Theorem 4.3 with $H = Q$.

Corollary 4.6 *Let $(Y, \{\varphi_N\})$ be a FC-space and Z be a topological space. Let $T \in KKM(Y, Z)$ be a compact mapping and $Q: Z \to 2^Y$ be set-valued mappings such that*

(1) *Q^{-1} is transfer compactly open-valued and for each $z \in Z$, $Q(z) \neq \emptyset$,*
(2) *for each $z \in Z$, $Q(z)$ is a FC-subspace of Y.*

Then there exists $(y_0, z_0) \in Y \times Z$ such that $z_0 \in T(y_0)$ and $y_0 \in Q(z_0)$.

Proof The conclusion of Corollary 4.6 holds from Corollary 4.5 with $P = Q$.

Remark 4.3 Theorem 4.3, Corollaries 4.5 and 4.6 generalize Theorem 2.5 of Lin et al. [14] from topological vector spaces to *FC*-spaces under much weak assumptions.

When $T \in KKM(Y, Z)$ is not compact, we have the following result.

Theorem 4.4 *Let $(Y, \{\varphi_N\})$ be a FC-space and Z be a topological space. Let $T \in KKM(Y,Z)$ and $H, P, Q\colon Z \to 2^Y$ be set-valued mappings such that*

(1) *Q^{-1} is transfer compactly open-valued and for each $z \in Z$, $Q(z) \neq \emptyset$,*
(2) *for each $z \in Z$, $Q(z) \subset H(z)$,*
(3) *for each $z \in Z$, $P(z)$ is a FC-subspace of Y relative to $H(z)$.*
(4) *for each compact subset D of Y, $\overline{T(D)}$ is compact in Z,*
(5) *there exists a compact subset K of Z such that for each $N \in \langle Y\rangle$, there is a compact FC-subspace L_N of Y containing N satisfying*

$$T(L_N)\setminus K \subset \bigcup_{y \in L_N} \mathrm{c\,int} Q^{-1}(y).$$

Then there exists $(y_0, z_0) \in Y \times Z$ such that $z_0 \in T(y_0)$ and $y_0 \in P(z_0)$.

Proof Since K is a compact subset of Z, Q has nonempty values and Q^{-1} is transfer compactly open-valued by (1), it follows from Lemma 2.1 that

$$K = \bigcup_{y \in Y} \left(\mathrm{c\,int} Q^{-1}(y) \bigcap K\right).$$

Hence there exists $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ such that

$$K = \bigcup_{i=0}^{n} \left(\mathrm{c\,int} Q^{-1}(y_i) \bigcap K\right) \subset \bigcup_{i=0}^{n} \mathrm{c\,int} Q^{-1}(y_i).$$

By (5), there exists a compact FC-subspace L_N of Y containing N satisfying

$$T(L_N)\setminus K \subset \bigcup_{y \in L_N} \mathrm{c\,int} Q^{-1}(y).$$

Hence, we have

$$T(L_N) = \bigcup_{y \in L_N} \left(\mathrm{c\,int} Q^{-1}(y) \bigcap T(L_N)\right) = \bigcup_{y \in L_N} \mathrm{int}_{T(L_N)} \left(Q^{-1}(y) \bigcap T(L_N)\right).$$

Since L_N is a compact FC-subspace of Y, $\overline{T(L_N)}$ is compact by (4). Since $T \in KKM(Y,Z)$ and L_N is a FC-subspace of Y, by Lemma 2.4, the restriction $T|_{L_N}$ of T on L_N is such that $T|_{L_N} \in KKM(L_N, Z)$ is compact. Define set-valued mappings $H_1, P_1, Q_1\colon T(L_N) \to 2^{L_N}$ by

$$H_1(z) = H(z) \bigcap L_N, \;\; P_1(z) = P(z) \bigcap L_N \quad \text{and} \quad Q_1(z) = Q(z) \bigcap L_N,$$
$$\forall \; z \in T(L_N).$$

It is easy to check that T_{L_N}, H_1, P_1, Q_1 satisfy all conditions of Theorem 4.3. Hence, there exist $z_0 \in T(L_N) \subset Z$ and $y_0 \in L_N \subset Y$ such that $z_0 \in T|_{L_N}(y_0) = T(y_0)$ and $y_0 \in P_1(z_0) \subset P(z_0)$. This completes the proof.

Corollary 4.7 *Let $(Y, \{\varphi_N\})$ be a FC-space and Z be a topological space. Let $T \in KKM(Y,Z)$ and $P, Q\colon Z \to 2^Y$ be set-valued mappings such that*

(1) *Q^{-1} is transfer compactly open-valued and for each $z \in Z$, $Q(z) \neq \emptyset$,*
(2) *for each $z \in Z$, $P(z)$ is a FC-subspace of Y relative to $Q(z)$.*
(3) *for each compact subset D of Y, $\overline{T(D)}$ is compact in Z,*

(4) *there exists a compact subset K of Z such that for each $N \in \langle Y \rangle$, there is a compact FC-subspace L_N of Y containing N satisfying*

$$T(L_N)\backslash K \subset \bigcup_{y \in L_N} \mathrm{c\,int} Q^{-1}(y).$$

Then there exists $(y_0, z_0) \in Y \times Z$ such that $z_0 \in T(y_0)$ and $y_0 \in P(z_0)$.

Proof The conclusion of Corollary 4.7 holds from Theorem 4.4 with $H = Q$.

Corollary 4.8 *Let $(Y, \{\varphi_N\})$ be a FC-space and Z be a topological space. Let $T \in KKM(Y, Z)$ and $Q\colon Z \to 2^Y$ be set-valued mappings such that*

(1) *Q^{-1} is transfer compactly open-valued and for each $z \in Z$, $Q(z) \neq \emptyset$,*
(2) *for each $z \in Z$, $Q(z)$ is a FC-subspace of Y.*
(3) *for each compact subset D of Y, $\overline{T(D)}$ is compact in Z,*
(4) *there exists a compact subset K of Z such that for each $N \in \langle Y \rangle$, there is a compact FC-subspace L_N of Y containing N satisfying*

$$T(L_N)\backslash K \subset \bigcup_{y \in L_N} \mathrm{c\,int} Q^{-1}(y).$$

Then there exists $(y_0, z_0) \in Y \times Z$ such that $z_0 \in T(y_0)$ and $y_0 \in P(z_0)$.

Proof The conclusion of Corollary 4.8 holds from Corollary 4.7 with $P = Q$.

Remark 4.4 Theorem 4.4, Corollaries 4.7 and 4.8 generalize Theorem 2.6 of Lin et al. [14] from topological vector spaces to *FC*-spaces under weaker assumptions.

Theorem 4.5 *Let $(Y, \{\varphi_N\})$ be a FC-space and Z be a topological space. Let $T \in KKM(Y, Z)$ and A, B, M be subsets of $Z \times Y$ such that*

(1) *The mapping $Q^{-1}\colon Y \to 2^Z$ is transfer compactly open-valued, where $Q\colon Z \to 2^Y$ is defined by $Q(z) = \{y \in Y : (z, y) \notin A\}$ for all $z \in Z$,*
(2) *for all $y \in Y$ and $z \in T(y)$, $(z, y) \in M$ and $B \subset A$,*
(3) *for each $z \in Z$, the set $\{y \in Y : (z, y) \notin M\}$ is a FC-subspace of Y relative to the set $\{y \in Y : (z, y) \notin B\}$.*
(4) *for each compact subset D of Y, $\overline{T(D)}$ is compact in Z,*
(5) *there exists a compact subset K of Z such that for $N \in \langle Y \rangle$, there is a compact FC-subspace L_N of Y containing N satisfying*

$$T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{z \in Z : (z, y) \in A\} \right) \subset K.$$

Then there exists a point $\hat{z} \in Z$ such that $\{\hat{z}\} \times Y \subset A$.

Proof Suppose the conclusion is false, then for each $z \in Z$, there exists a point $y \in Y$ such that $(z, y) \notin A$ and hence for each $z \in Z$, $Q(z) \neq \emptyset$. Define set-valued mapping $H, P\colon Z \to 2^Y$ by

$$H(z) = \{y \in Y : (z, y) \notin B\} \quad \text{and} \quad P(z) = \{y \in Y : (z, y) \notin M\}, \quad \forall\ z \in Z.$$

Since $B \subset A$ by (2), we have $Q(z) \subset H(z)$ for each $z \in Z$. By (3) and the definition of H and P, for each $z \in Z$, $P(z)$ is a FC-subspace of Y relative to $H(z)$. By (4), we have

$$\begin{aligned} T(L_N)\backslash K &\subset T(L_N)\backslash \left(T(L_N) \bigcap \Big(\bigcap_{y \in L_N} \mathrm{ccl}\{z \in Z : (z,y) \in A\}\Big) \right) \\ &\subset Z \backslash \bigcap_{y \in L_N} \mathrm{ccl}\{z \in Z : (z,y) \in A\} \\ &= \bigcup_{y \in L_N} \mathrm{c\,int}\{z \in Z : (z,y) \notin A\} = \bigcup_{y \in L_N} \mathrm{c\,int} Q^{-1}(y). \end{aligned}$$

All conditions of Theorem 4.4 are satisfied. By Theorem 4.4, there exists $(y_0, z_0) \in Y \times Z$ such that $z_0 \in T(y_0)$ and $y_0 \in P(z_0)$. It follows that $z_0 \in T(y_0)$ and $(z_0, y_0) \notin M$ which contradicts the condition (2). Hence there exists $\hat{z} \in Z$ such that $\{\hat{z}\} \times Y \subset A$.

Corollary 4.9 *Let $(Y, \{\varphi_N\})$ be a FC-space and Z be a topological space. Let $T \in KKM(Y,Z)$ and $A \subset Z \times Y$ such that*

(1) *The mapping $Q^{-1}\colon Y \to 2^Z$ is transfer compactly open-valued, where $Q\colon Z \to 2^Y$ is defined by $Q(z) = \{y \in Y : (z,y) \notin A\}$ for all $z \in Z$,*
(2) *for all $y \in Y$ and $z \in T(y)$, $(z,y) \in A$,*
(3) *for each $z \in Z$, the set $\{y \in Y : (z,y) \notin A\}$ is a FC-subspace of Y,*
(4) *for each compact subset D of Y, $\overline{T(D)}$ is compact in Z,*
(5) *there exists a compact subset K of Z such that for each $N \in \langle Y \rangle$, there is a compact FC-subspace L_N of Y containing N satisfying*

$$T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{z \in Z : (z,y) \in A\} \right) \subset K.$$

Then there exists a point $\hat{z} \in Z$ such that $\{\hat{z}\} \times Y \subset A$.

Proof The conclusion of Corollary 4.9 holds from Theorem 4.5 with $A = B = M$.

Remark 4.5 Theorem 4.5 and Corollary 4.9 generalize Theorem 2.8 of Lin et al. [14] from topological vector spaces to FC-spaces under weaker assumptions.

In order to distinguishing our results from others, now we state the following special cases of our results.

Theorem 4.6 *Let Y be a convex space and Z be a topological space. Let $T \in KKM(Y,Z)$ and $H, P, Q\colon Z \to 2^Y$ be set-valued mappings such that*

(1) *Q^{-1} is a transfer compactly open-valued mapping,*
(2) *for each $z \in Z$, $Q(z) \neq \emptyset$ and $Q(z) \subset H(z)$,*
(3) *for each $z \in Z$ and each $N \in \langle H(z) \rangle$, $\mathrm{co}(N) \subset P(z)$.*
(4) *for each compact subset D of Y, $\overline{T(D)}$ is compact in Z,*
(5) *there exists a compact subset K of Z such that for each $N \in \langle Y \rangle$, there is a compact convex subset L_N of Y containing N satisfying*

$$T(L_N)\backslash K \subset \bigcup_{y \in L_N} \mathrm{c\,int} Q^{-1}(y).$$

Then there exists $(y_0, z_0) \in Y \times Z$ such that $z_0 \in T(y_0)$ and $y_0 \in P(z_0)$.

Proof For each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, define a mapping $\varphi_N \colon \Delta_n \to Y$ by

$$\varphi_n(\lambda) = \sum_{i=0}^{n} \lambda_i y_i, \quad \forall\, \lambda = (\lambda_0, \ldots, \lambda_n) \in \Delta_n.$$

Clearly, φ_N is continuous and $\varphi_N(\Delta_n) = \mathrm{co}(N)$. Hence (Y, φ_N) is a FC-space and each convex subset of Y is a FC-subspace of (Y, φ_N). The condition (3) implies that for each $z \in Z$, $P(z)$ is a FC-subspace of Y relative to $H(z)$. All conditions of Theorem 4.4 are satisfied. The conclusion of Theorem 4.6 holds from Theorem 4.4.

Corollary 4.10 *Let Y be a convex space and Z be a topological space. Let $T \in KKM(Y, Z)$ and $P \colon Z \to 2^Y$ be set-valued mappings such that*

(1) *P^{-1} is a transfer compactly open-valued mapping,*
(2) *for each $z \in Z$, $P(z)$ is nonempty convex,*
(3) *for each compact subset D of Y, $\overline{T(D)}$ is compact in Z,*
(4) *there exists a compact subset K of Z such that for each $N \in \langle Y \rangle$, there is a compact convex subset L_N of Y containing N satisfying*

$$T(L_N) \backslash K \subset \bigcup_{y \in L_N} \mathrm{c\,int} P^{-1}(y).$$

Then there exists $(y_0, z_0) \in Y \times Z$ such that $z_0 \in T(y_0)$ and $y_0 \in P(z_0)$.

Proof The conclusion of Corollary 4.10 holds from Theorem 4.6 with $P = Q = H$.

Remark 4.6 Corollary 4.10 generalizes Theorem 2.6 of Lin et al. [14] from P^{-1} being a transfer open-valued mapping to P^{-1} being a transfer compactly open-valued mapping. Hence Theorem 4.4 further generalizes Theorem 2.6 of Lin et al. [14] in following ways: (1) from convex spaces with linear structure to FC-spaces without any convexity structure; (2) from the transfer open-valued mapping to transfer compactly open-valued mapping; (3) the class KKM(Y, Z) in Theorem 4.4 includes the corresponding class KKM(Y, Z) in Theorem 2.6 of [14] as proper subclass; (4) from two set-valued mappings to four set-valued mappings.

Finally, we emphasis that the most of KKM type theorems, coincidence theorems and section theorems stated in topological vector spaces, H-spaces, G-convex spaces and L-convex spaces in previous works are all special cases of our results. In the study of optimization problems (vector) equilibrium problems and many nonlinear problems, the domain and range of mappings may not have the convexity structure. Our results in this paper can be applied to deal with these problems without convexity structure, but the corresponding results in topological vector spaces can not be applied. In fact, the results in this paper have been applied to establish some existence theorems of solutions for vector equilibrium problems in FC-spaces in a follow-up paper with same title.

Acknowledgements This project was supported by the NSF of Sichuan Education Department of China (2003A081) and SZD0406.

References

1. Knaster, H., Kuratowski, C., Mazurkiewicz, S.: Ein Beweis des Fixpunktsatzes f*ü*r n-dimensionale simplexe. Fundam. Math. **14**, 132–137 (1929)
2. Fan, Ky.: A generalization of Tychonoff's fixed point theorem. Math. Ann. **142**, 305–310 (1961)
3. Ding, X.P.: KKM type theorems and coincidence theorems on *K*-convex spaces. Acta. Math. Sci. **22**(3), 419–426 (2002)
4. Ding, X.P.: Generalized *G-KKM* theorems in generalized convex spaces and their applications. J. Math. Anal. Appl. **266**(1), 21–37 (2002)
5. Ding, X.P.: Generalized *L-KKM* type theorems in *L*-convex spaces with applications. Comput. Math. Appl. **43**, 1249–1256 (2002)
6. Ding, X.P.: *KKM* type theorems, minimax inequalities and saddle point theorems in generalized convex spaces. Acta. Math. Sin. (in Chinese) **47**(4), 711–722 (2004)
7. Ding, X.P.: Generalized *R-KKM* type theorems in topological spaces and applications. J. Sichuan Norm. Univ. (*NS*) **28**(5), 505–513 (2005)
8. Ding, X.P., Liou, Y.C., Yao, J.C.: Generalized *R-KKM* type theorems in topological spaces with applications. Appl. Math. Lett. **18**(12), 1345–1350 (2005)
9. Deng, L., Xia, X.: Generalized *R-KKM* theorems in topolgical space and their applications. J. Math. Anal. Appl. **285**, 679–690 (2003)
10. Chang, T.H., Yen, C.L.: *KKM* property and fixed point theorems. J. Math. Anal. Appl. **203**, 224–235 (1996)
11. Park, S., Singh, S.P., Watson, B.: Some fixed point theorems for composites of acyclic maps. Proc. Am. Math. Soc. **121**, 1151–1158 (1994)
12. Park, S., Kim, H.: Foundations of the *KKM* theory on generalized convex spaces. J. Math. Anal. Appl. **209**(2), 551–571 (1997)
13. Ben-El-Mechaiekh, H., Deguire, P.: Approachability and fixed point for non-convex set-valued maps. J. Math. Anal. Appl. **170**, 477–500 (1992)
14. Lin, L.J., Ansari, Q.H., Wu, J.Y.: Geometric properties and coincidence theorems with applications to generalized vector equilibrium problems. J. Optim. Theory Appl. **117**(1), 121–137 (2003)
15. Lin, L.J., Wan, W.P.: *KKM* type theorems and coincidence theorems with applications to the existence of equilibria. J. Optim. Theory Appl. **123**(1), 105–122 (2004)
16. Lin, L.J., Chen, H.L.: The study of *KKM* theorems with applications to vector equilibrium problems and implicit vector variational inequalities problems. J. Global Optim. **32**, 135–157 (2005)
17. Ding, X.P.: Coincidence theorems and equilibria of generalized games. Indian J. Pure Appl. Math. **27**(11), 1057–1071 (1996)
18. Lin, L.J.: System of coincidence theorems with applications. J. Math. Anal. Appl. **285**(2), 408–418 (2003)
19. Ding, X.P.: Coincidence theorems in topological spaces and their applications. Appl. Math. Lett. **12**, 99–105 (1999)
20. Ding, X.P.: Maximal element theorems in product *FC*-spaces and generalized games. J. Math. Anal. Appl. **305**(1), 29–42 (2005)
21. Horvath, C.D.: Contractibility and general convexity. J. Math. Anal. Appl. **156**(2), 341–357 (1991)
22. Ben-El-Mechaiekh, H., Chebbi, S., Flornzano, M., Llinares, J.V.: Abstract convexity and fixed points. J. Math. Anal. Appl. **222**, 138–150 (1998)
23. Chang, T.H., Huang, Y.Y., Jeng, J.C., Kuo, K.W.: On *S-KKM* property and related topies. J. Math. Anal. Appl. **229**, 212–227 (1999)
24. Lin, L.J.: Applications of a fixed theorem in *G*-convex space. Nonlinear Anal. **46**, 601–608 (2001)
25. Aubin, J.P., Ekeland, I.: Applied nonlinear analysis. Wiley, New York (1984)
26. Ha, C.W.: Minimax and fixed-point theorems. Math. Annal. **248**, 73–77 (1980)